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A Generalization of Chebyshev Polynomials

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1. INTRODUCTION

Let $\nu_1, ..., \nu_n$ be fixed natural numbers. Set $N = \nu_1 + \cdots + \nu_n - 1$. Construct Newton's interpolation formula

$$f(x) = L_N(f; x) + R_N(x)$$

on the basis of the nodes $(x_k)_1^n$, $a \leqslant x_1 < \cdots < x_n \leqslant b$, with multiplicities $(\nu_k)_1^n$ respectively. It is a well-known fact that

$$\| R_N \| \leqslant \| f^{(N+1)} \| \, \| (x - x_1)^{\nu_1} \cdots (x - x_n)^{\nu_n} \|$$

where $\| g \| := \max\{| g(x)|: a \leqslant x \leqslant b\}$. So, the extremal problem:

$$\text{determine} \inf_{x_1 < \cdots < x_n} \| (x - x_1)^{\nu_1} \cdots (x - x_n)^{\nu_n} \| \tag{1.1}$$

is a natural question suggested by the above mentioned classical interpolation process. The solution of (1.1) in the simple case $\nu_1 = \cdots = \nu_n = 1$ leads to the famous Chebyshev polynomials of the first kind.

The purpose of our paper is to prove the existence and uniqueness of extremal nodes $(x^*)_1^n$ in problem (1.1) for any fixed system of multiplicities $(\nu_k)_1^n$. As an auxiliary result we give a multiple nodes extension of a theorem of Davis [1] (see also [2], [3], [4]) concerning interpolation at extremal points for algebraic polynomials.

Note that L. Tschakaloff [5] (see also Popoviciu [6]) has arrived at the same problem (1.1) but with $\| \cdot \| = \| \cdot \|_{L_2[a,b]}$ studying mechanical quadratures of highest degree of precision. He proved the existence of extremal nodes for this case. The uniqueness, remaining an open problem for 20 years, was established recently by Ghizzetti and Ossicini [7]. An extension of this L_2-problem was considered in a paper of Karlin and Pinkus [8].

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2. Interpolation at Extremal Points

We start with an auxiliary proposition.

THEOREM 1. *Let $(\nu_k)_1^n$ be arbitrary fixed natural numbers and let $p(x)$ be a continuous function defined and $\geqslant 0$ on $[x_0, \infty)$, having a finite number of zeros in any finite subinterval $[x_0, x]$. Given positive numbers $(e_k)_{k=1}^n$, there exists a unique system of points $(x_k)_1^n$, $x_0 < x_1 < \cdots < x_n$, such that*

$$\left| \int_{x_{k-1}}^{x_k} p(x) \prod_{i=1}^{n} (x - x_i)^{\nu_i} \, dx \right| = e_k, \ k = 1,..., n.$$

Proof. Define

$$\varphi_k(x_1,..., x_n) = \left| \int_{x_{k-1}}^{x_k} p(x) \prod_{i=1}^{n} (x - x_i)^{\nu_i} \, dx \right| - e_k, \ k = 1,..., n.$$

Let $J(x_1,..., x_n; p(x))$ denote the Jacobian

$$\frac{D(\varphi_1,..., \varphi_n)}{D(x_1,..., x_n)} = \begin{vmatrix} \int_{x_0}^{x_1} p(x)\,\omega(x)\,\omega_1(x)\,dx & \cdots & \int_{x_0}^{x_1} p(x)\,\omega(x)\,\omega_n(x)\,dx \\ \int_{x_1}^{x_2} p(x)\,\omega(x)\,\omega_1(x)\,dx & \cdots & \int_{x_1}^{x_2} p(x)\,\omega(x)\,\omega_n(x)\,dx \\ \cdots & \cdots & \cdots \\ \int_{x_{n-1}}^{x_n} p(x)\,\omega(x)\,\omega_1(x)\,dx & \cdots & \int_{x_{n-1}}^{x_n} p(x)\,\omega(x)\,\omega_n(x)\,dx \end{vmatrix}$$

where $\omega_i(x) = -\epsilon_i \nu_i (x - x_1) \cdots (x - x_n)/(x - x_i)$, $i = 1, 2,..., n$,

$$\epsilon_i = \operatorname{sign} \prod_{k=1}^{n} \left(\frac{x_i + x_{i-1}}{2} - x_k \right)^{\nu_k},$$

and

$$\omega(x) = \prod_{i=1}^{n} (x - x_i)^{\nu_i - 1}.$$

We claim that $J(x_1,..., x_n; p(x)) \neq 0$ provided $x_0 < x_1 < \cdots < x_n$. Indeed, otherwise there exist real numbers $(b_i)^n$ such that $\sum_{i=1}^n |b_i| > 0$ and

$$\int_{x_{k-1}}^{x_k} p(x)\,\omega(x)\{b_1\omega_1(x) + \cdots + b_n\omega_n(x)\}\,dx = 0$$

for $k = 1,\dots, n$. Clearly the polynomial $Q(x) = b_1\omega_1(x) + \cdots + b_n\omega_n(x)$ must change sign at least once in (x_{k-1}, x_k). Therefore $Q(x)$ has n sign changes in (x_0, x_n). But $Q \in \pi_{n-1}$ (π_m denotes the class of all algebraic polynomials of degree $\leqslant m$).

The proof of Theorem 1 proceeds by induction on n. Let $n = 1$. Clearly, the function

$$f(\xi) = \left| \int_{x_0}^{\xi} p(x)(x-\xi)^{\nu_1}\, dx \right|$$

is strictly increasing and $f(x_0) = 0$. Therefore there exists only one number $x_1 > x_0$ with $f(x_1) = e_1$. Now suppose that the theorem holds for every choice of the multiplicities $(\mu_i)_1^{n-1}$, the values $(e_i)_1^{n-1}$ and the weight $p(x)$. Then the problem

$$\left(\begin{matrix} e_1, \dots, e_{n-2}, e_{n-1} \\ \nu_1, \dots, \nu_{n-2}, \nu_{n-1} + \nu_n \end{matrix} \middle| \, p(x) \right) \tag{2.1}$$

(i.e., the problem of Theorem 1 with the noted parameters) has a unique solution $(t_k)_1^{n-1}$ in (x_0, ∞). Let $\xi \geqslant x_0$. Denote by $\{x_k(\xi)\}_{k=1}^{n-1}$ the unique solution of the problem

$$\left(\begin{matrix} e_1, \dots, e_{n-1} \\ \nu_1, \dots, \nu_{n-1} \end{matrix} \middle| \, p_\xi(x) \right), \quad p_\xi(x) = p(x) |\, x - \xi\,|^{\nu_n}.$$

Since $J(x_1(\xi),\dots, x_{n-1}(\xi); p_\xi(x)) \neq 0$, it follows from the implicit function theorem that $x_k(\xi)$, $k = 1,\dots, n$, are continuous functions in (x_0, ∞). Now we shall show that

$$x_{n-1}(\xi) < \xi \qquad \text{for} \quad \xi > t_{n-1}. \tag{2.2}$$

Recall that $x_{n-1}(t_{n-1}) = t_{n-1}$. Suppose that $x_{n-1}(\xi_0) = \xi_0$ for some $\xi_0 > t_{n-1}$. Then the problem (2.1) would have two different solutions: $(t_k)_1^{n-1}$ and $\{x_k(\xi_0)\}_1^{n-1}$, which is impossible by the induction hypothesis. Therefore $x_{n-1}(\xi) - \xi \neq 0$ for every $\xi > t_{n-1}$. Thus, in order to prove (2.2) we need only to demonstrate that $x_{n-1}(\xi) < \xi$ for sufficiently large ξ. We shall show even more, namely, that $x_{n-1}(\xi)$ is bounded in $[x_0, \infty)$. Indeed, suppose that $\limsup_{\xi\to\infty} x_{n-1}(\xi) = \infty$. Then there exist an index k, $1 \leqslant k \leqslant n-1$, and a point $\alpha > x_0$ such that

$$\begin{gathered} \limsup_{\xi\to\infty} x_k(\xi) = \infty \\ \text{and } x_{k-1}(\xi) \leqslant \alpha \text{ for } \xi \in [x_0, \infty). \end{gathered} \tag{2.3}$$

Clearly,

$$\begin{aligned} e_k &\geqslant (\xi - (\alpha+1))^{\nu_n} \prod_{i=k}^{n-1} \{x_i(\xi) - (\alpha+1)\}^{\nu_i} \int_\alpha^{\alpha+1} p(x)(x-\alpha)^{\nu_1+\cdots+\nu_{k-1}}\, dx \\ &=: r(\xi) \end{aligned} \tag{2.4}$$

for every ξ such that $\min\{\xi, x_k(\xi)\} \geqslant \alpha + 1$. But $\limsup_{\xi\to\infty} r(\xi) = \infty$ in view of (2.3). Thus $x_{n-1}(\xi) < \xi$ for sufficiently large ξ and (2.2) follows.

Consider the function

$$g(\xi) = \left| \int_{x_{n-1}(\xi)}^{\xi} p(x)(\xi - x)^{\nu_n} \prod_{i=1}^{n-1} (x - x_i(\xi))^{\nu_i} \, dx \right| .$$

It is continuous in $[t_{n-1}, \infty)$ and $g(t_{n-1}) = 0$. One can see, as in (2.4), that $\limsup_{\xi\to\infty} g(\xi) = \infty$. Hence there exists a point x_n such that $g(x_n) = e_n$ and $x_n > t_{n-1}$. It remains to show uniqueness of x_n. Let us assume that $g(\xi_1) = g(\xi_2) = e_n$ and $t_{n-1} < \xi_1 < \xi_2$. By Rolle's theorem there exists a point $\eta \in (\xi_1, \xi_2)$ for which $g'(\eta) = 0$. But

$$\begin{aligned} g'(\eta) &= \frac{\partial}{\partial \xi} \varphi_n(x_1(\xi),..., x_{n-1}(\xi), \xi)|_{\xi=\eta} \\ &= \frac{J(x_1(\eta),..., x_{n-1}(\eta), \eta; p(x))}{J(x_1(\eta),..., x_{n-1}(\eta); p_n(x))} . \end{aligned}$$

Hence $J(x_1(\eta),..., x_{n-1}(\eta), \eta; p(x)) = 0$, which contradicts our previous observation. The theorem is proved.

COROLLARY 1. *Let $(\nu_k)_1^n$ be a fixed system of arbitrary natural numbers. Let the real numbers $(y_k)_0^{n+1}$ satisfy the requirements $y_k \neq y_{k-1}$, $k = 1,..., n + 1$, and*

$$\begin{aligned} | y_{k-1} - y_k | + | y_k - y_{k+1} | &= | y_{k-1} - y_{k+1} | \quad \text{if } \nu_k \text{ is even,} \\ | y_{k-1} - y_k | + | y_k - y_{k+1} | &> | y_{k-1} - y_{k+1} | \quad \text{if } \nu_k \text{ is odd,} \end{aligned}$$

$k = 1,..., n$. Given an interval $[a, b]$, there exists a unique polynomial $P \in \pi_N$, $N = \nu_1 + \cdots + \nu_n + 1$, and a unique system of points $(x_k)_1^n$, $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$, such that

$$\begin{aligned} P(x_k) &= y_k , \; k = 0, 1,..., n + 1, \\ P^{(\lambda)}(x_k) &= 0, \; k = 1,..., n, \; \lambda = 1,..., \nu_k . \end{aligned} \tag{2.5}$$

Proof. Denote by $(t_k)_1^n$ the unique solution of the problem

$$\binom{| y_1 - y_0 | , | y_2 - y_1 | ,..., | y_n - y_{n-1} |}{\nu_1 , \nu_2 ,..., \nu_n}$$

in $[a, \infty)$ (i.e., the problem of Theorem 1 with weight $p(x) = 1$). The polynomial $P_1(t) = (t - t_1)^{\nu_1} \cdots (t - t_n)^{\nu_n}$ is strictly monotone in $[t_n, \infty)$.

Choose the point $\beta > t_n$ by the condition $| y_{n+1} - y_n | = \int_{t_n}^{\beta} | P_1(t)| \, dt$. Set

$$\tilde{P}(t) = y_0 + \operatorname{sign}(y_1 - y_0) \int_a^t P_1(\tau) \, d\tau.$$

It is easily seen that $x_k = a + [(b - a)/(\beta - a)](t_k - a)$, $k = 1,..., n$, and that $P(x) = \tilde{P}(a + [(\beta - a)/(b - a)](x - a))$ is the unique polynomial from π_N satisfying (2.5).

Remark 1. The particular case $\nu_1 = \cdots = \nu_n = 1$ of Corollary 1 was studied by Davis [1], Miczelski and Paszkowski [2], Videnskiĭ [3], and Fitzgerald and Schumaker [4].

The following is an immediate consequence of Corollary 1.

COROLLARY 2. *Let $(\nu_k)_1^n$ be a fixed system of arbitrary odd natural numbers. There exists a unique polynomial $\tau_N(\boldsymbol{\nu}; x)$ of degree $N = \nu_1 + \cdots + \nu_n + 1$ and leading coefficient* 1 *and a unique system of points $(x_k)_1^n$, $-1 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1$, such that*

$$\tau_N(\boldsymbol{\nu}; x_k) = (-1)^{n+1-k} \| \tau_N(\boldsymbol{\nu}); \cdot) \|_{C[-1,1]}, \; k = 0, 1,..., n + 1,$$

and

$$\tau_N^{(\lambda)}(\boldsymbol{\nu}; x_k) = 0, \; \lambda = 1,..., \nu_k .$$

Clearly $\tau_N(\boldsymbol{\nu}; x) = \cos(N \arccos x)$ for $| x | \leqslant 1$ in the case $\nu_1 = \cdots = \nu_n = 1$. Thus $\tau_N(\boldsymbol{\nu}; x)$ can be considered as a generalizations of the Chebyshev polynomial of the first kind. In the next section we give another extension of this classical polynomial.

3. MAIN RESULT

Let the multiplicities $(\nu_k)_1^n$ and the point a be fixed. Suppose the numbers $(e_k)_1^{n+1}$ are positive. By virtue of Theorem 1 there exists a unique system of points $a = x_0 < x_1 < \cdots < x_n < x_{n+1}$ such that

$$\left| \int_{x_{k-1}}^{x_k} W(x) \, dx \right| = e_k, \; k = 1,..., n + 1,$$

where $W(x) = (x - x_1)^{\nu_1} \cdots (x - x_n)^{\nu_n}$. We shall use in the sequel the following property of the last point x_{n+1}.

LEMMA 1. *The point x_{n+1} is a differentiable function of $e_1, ..., e_{n+1}$ in the domain $G := \{(e_1, ..., e_{n+1}): e_k > 0, k = 1,..., n + 1\}$. Moreover $\partial x_{n+1}/\partial e_k > 0$ $(k = 1,..., n + 1)$ in G.*

Proof. Set $\epsilon_{n+1} = 1$, $\epsilon_k = (-1)^{\nu_k} \epsilon_{k+1}$, $k = 1,..., n$. We proved in Theorem 1 that the functions $\{x_k(e_1, ..., e_{n+1})\}_1^{n+1}$ are uniquely defined by the system of equations

$$\varphi_k(x_1, ..., x_{n+1}; e_1, ..., e_{n+1}) := \int_{x_{k-1}}^{x_k} W(x)\, dx - \epsilon_k e_k = 0, k = 1,..., n + 1.$$

Let us abbreviate $J(x_1, ..., x_n; p(x))$ to $J(x_1, ..., x_n)$ in case $p(x) = 1$. Since

$$\frac{D(\varphi_1, ..., \varphi_{n+1})}{D(x_1, ..., x_{n+1})} = W(x_{n+1})\, J(x_1, ..., x_n) \neq 0,$$

we conclude on the basis of the implicit function theorem that $\{x_k(e_1, ..., e_{n+1})\}_1^{n+1}$ are differentiable functions in G. It is not difficult to verify that

$$\frac{\partial x_{n+1}}{\partial e_k} = \frac{D(\varphi_1, ..., \varphi_n, \varphi_{n+1})}{D(x_1, ..., x_n, e_k)} : \frac{D(\varphi_1, ..., \varphi_n, \varphi_{n+1})}{D(x_1, ..., x_n, x_{n+1})}$$

$$= \frac{(-1)^{k+1} \epsilon_k}{W(x_{n+1})\, J(x_1, ..., x_n)} \begin{vmatrix} \int_{x_0}^{x_1} \omega(x)\, \omega_1(x)\, dx & \cdots & \int_{x_0}^{x_1} \omega(x)\, \omega_n(x)\, dx \\ \cdots & \cdots & \cdots \\ \int_{x_{k-2}}^{x_{k-1}} \omega(x)\, \omega_1(x)\, dx & \cdots & \int_{x_{k-2}}^{x_{k-1}} \omega(x)\, \omega_n(x)\, dx \\ \int_{x_k}^{x_{k+1}} \omega(x)\, \omega_1(x)\, dx & \cdots & \int_{x_k}^{x_{k+1}} \omega(x)\, \omega_n(x)\, dx \\ \cdots & \cdots & \cdots \\ \int_{x_n}^{x_{n+1}} \omega(x)\, \omega_1(x)\, dx & \cdots & \int_{x_n}^{x_{n+1}} \omega(x)\, \omega_n(x)\, dx \end{vmatrix}$$

As in the proof of Theorem 1 one can show that the last determinant does not vanish in G. Since $W(x_{n+1}) > 0$ and $J(x_1, ..., x_n) \neq 0$ in G, we conclude that $\partial x_{n+1}/\partial e_k$ has a constant sign in G. Therefore x_{n+1} is a strictly monotone function with respect to e_k. On the other hand

$$x_{n+1}(e_1, ..., e_{k-1}, e_k, e_{k+1}, ..., e_{n+1}) > x_{n+1}(e_1, ..., e_{k-1}, 1, e_{k+1}, ..., e_{n+1})$$

for sufficiently large e_k. Hence x_{n+1} is a monotone increasing function of e_k, $k = 1,..., n + 1$. The lemma is proved.

THEOREM 2. *Let* $(\nu_k)_1^n$ *be a fixed system of arbitrary natural numbers. Given* $[a, b]$, *there exists a unique system of points* $(x^*)_1^n$ *such that*

$$\| (x - x_1^*)^{\nu_1} \cdots (x - x_n^*)^{\nu_n} \| = \inf_{a \leqslant x_1 < \cdots < x_n \leqslant b} \| (x - x_1)^{\nu_1} \cdots (x - x_n)^{\nu_n} \| .$$

Moreover, $a < x_1^* < \cdots < x_n^* < b$. *The extremal polynomial* $T_m(\boldsymbol{\nu}; x) = (x - x_1^*)^{\nu_1} \cdots (x - x_n^*)^{\nu_n}$ $(m = \nu_1 + \cdots + \nu_n)$ *is uniquely determined by the condition that there exist* $n - 1$ *points* $(t_k)_1^{n-1}$, $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, *such that*

$$T_m(\boldsymbol{\nu}; t_k) = (-1)^{m - \nu_1 - \cdots - \nu_k} \| T_m(\boldsymbol{\nu}; \cdot) \|$$

for $k = 0, 1,..., n$.

Proof. Acording to Corollary 1, there exists a unique polynomial $P \in \pi_m$ and a unique system of points

$$a = t_0 < x_1^* < t_1 < \cdots < t_{n-1} < x_n^* < t_n = b$$

such that

$$P(t_k) = (-1)^{m - \nu_1 - \cdots - \nu_k}, \; k = 0,..., n,$$

$$P'(t_k) = 0, \; k = 1,..., n - 1,$$

and

$$P^{(\lambda)}(x_k^*) = 0, \; k = 1,..., n, \; \lambda = 0,..., \nu_k - 1.$$

Evidently $\| P \| = 1$ since $P'(x)$ vanishes only at $(x^*)_1^n$ and $(t_k)_1^{n-1}$. It is clear that

$$P(x) = C(x - x_1^*)^{\nu_1} \cdots (x - x_n^*)^{\nu_n}, \; C = 1/\{(b - x_1^*)^{\nu_1} \cdots (b - x_n^*)^{\nu_n}\}.$$

Denote $T_m(\boldsymbol{\nu}; x) = C^{-1} \cdot P(x)$. We shall show that $T_m(\boldsymbol{\nu}; x)$ is the desired polynomial. Indeed, let us assume that there is a polynomial Q of the form $Q(x) = (x - x_1)^{\nu_1} \cdots (x - x_n)^{\nu_n}$ with $| x_1 - x_1^* | + \cdots + | x_n - x_n^* | > 0$ such that

$$\| Q \| < \| T_m(\nu; \cdot) \| =: E. \tag{3.1}$$

Clearly $Q'(x)$ has $q \leqslant 2n - 1$ distinct real zeros. Let us denote them by $(\theta_k)_1^q$, $a = \theta_0 < \theta_1 < \cdots < \theta_q < \theta_{q+1} = b$. It follows from (3.1) that

$$e_k := | Q(\theta_k) - Q(\theta_{k-1}) | < E,$$

if θ_k or $\theta_{k-1} \in \{x_i\}_1^n$, and $<2E$ otherwise, for $k = 1,..., q$. Hence Lemma 1 implies $\theta_{q+1} < t_n$, i.e., $b < b$, a contradiction. The theorem is proved.

Remark 2. In the case $\nu_1 = \cdots = \nu_n = \mu$ it can be shown directly that

$$\min_{x_1 < \cdots < x_n} \| (x - x_1)^{\nu_1} \cdots (x - x_n)^{\nu_n} \|_{C[-1,1]}$$

$$= \left\| \left\{ \frac{1}{2^{n-1}} \cdot \cos(n \arccos x) \right\}^{\mu} \right\|_{C[-1,1]}$$

$$= 1/2^{(n-1)\mu}.$$

Finally, we make two conjuctures:

1. $| P(x) | \leqslant | T_m(\nu; x) |$ if $x \notin (a, b)$;
2. $\| P^{(k)} \|_{C[a,b]} \leqslant T_m^{(k)}(\nu; b)$, $k = 0, \ldots, m$,

for every polynomial $P(x)$ of the form

$$P(x) = A(x - x_1)^{\nu_1} \cdots (x - x_n)^{\nu_n}, \; x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n ,$$

A = const, such that

$$\| P \|_{C[a,b]} \leqslant \| T_m(\nu; \cdot) \|_{C[a,b]} .$$

References

1. C. Davis, Problem 4714, *Amer. Math. Monthly* **63** (1956), 729; Solution, *Amer. Math. Monthly* **64** (1957), 679–680.
2. J. Mycielski and S. Paszkowski, A generalization of Tschebysheff polynomials, *Bull. Acad. Polon. Sci. Ser. Math. Astron. Phys.* **8** (1960), 433–438.
3. V. S. Videnskiĭ, On a certain class of interpolation polynomials with non-fixed nodes, *Dokl. Akad. Nauk SSSR* **162** (1965), 251–254.
4. C. Fitzgerald and L. L. Schumaker, A differential equation approach to interpolation at extremal points, *J. Analyse Math.* **22** (1969), 117–134.
5. L. Tschakaloff, General quadrature formulae of Gaussian type, *Bulgar. Akad. Nauk Izv. Mat. Inst.* **1, 2** (1954), 67–84.
6. T. Popoviciu, Asurpa unei generalizari a formulei de integrare numerica a lui Gauss, *Acad. R. P. Romine Fil. Iasi Stud. Cerc. Sti.* **6** (1955), 29–57.
7. A. Ghizzetti and A. Ossicini, Sull' esistenza e unicita delle formule di quadratura gaussiane, *Rend. Mat.* (1) **8** (1975), 1–15.
8. S. Karlin and A. Pinkus, An extremal property of multiple Gaussian nodes, *in* "Studies in Spline Functions and Approximation Theory," pp. 143–162, Academic Press, New York, 1976.